

STRUCTURE IN SIMPLEXES. III. COMPOSITION SERIES⁽¹⁾

BY

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The Jacobson structure space has played a central role in the algebraic classification of C^* -algebras. Kaplansky showed [9] that for GCR algebras, one may find an ascending transfinite "composition series" of open sets G_α exhausting the structure space, such that $G_{\alpha+1} - G_\alpha$ is Hausdorff, and the component algebra over $G_{\alpha+1} - G_\alpha$ is homogeneous. The latter algebras may be characterized as continuous sections of certain fiber bundles (see [6, §10]); hence the classification of GCR algebras is reduced to finding how homogeneous algebras may be "put together".

An analogous structure space was defined for Choquet simplexes, and more generally for simplex spaces in [7] and [8]. In this paper we shall investigate composition series for simplex spaces. The most promising candidates for the components of such a composition series are the " C -spaces", the simplex spaces of the form $C_0(X)$, where X is a locally compact Hausdorff space. We say that a simplex space is a " GC -space" if it has such a composition series. We have succeeded in characterizing the separable GC -spaces in Theorem 5.1 as the separable simplex spaces for which the extremal states form an F_σ subset of the states, and closed subsets of the structure space are second category in themselves. By constructing various M -spaces in §7, we show that both of these conditions are necessary, even for this restricted class of simplex spaces.

§1 is devoted to several preliminary results concerning closed ideals and quotients of simplex spaces. In §2 we define a " C -ideal" in a simplex space to be a closed ideal which is itself a C -space. In Theorem 2.2 we show that if J is such an ideal in a simplex space A , then points in $\max J$ may be separated from distinct points in $\max A$ by disjoint open sets. This and other results are false (see §7.1) when one assumes only that J is an M -space, or equivalently (see Corollary 2.6) that $\max J$ is Hausdorff. To avoid this difficulty, we define an " M -ideal" J to be a closed ideal for which $\max J$ is "separated" in $\max A$. Equivalent conditions are given in Theorem 2.5.

In §§3 and 4 we consider a general theory of composition series. The reader may wish to avoid the somewhat tedious and routine proofs of these sections on a first reading of the paper. The theory is applicable when the components are C -ideals, M -ideals, or M -ideals with locally compact structure. The simplex spaces which

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are exhausted by such series are called *GC*-, *GM*-, and *GMLC*-spaces, respectively. These are distinct classes even for the separable *M*-spaces (see §7).

In §5 we prove the main theorems of the paper. Separable *GC*- and *GMLC*-spaces are characterized in Theorems 5.1 and 5.2. Necessary and sufficient conditions for a *GM*-space to be *GMLC* are given in Theorem 5.3.

In §6 we give a simplex space analogue of the Bishop-de Leeuw peaking description of the Choquet boundary of a function algebra. This in part generalizes a result of Rogalski for simplex spaces with "good spectrum" (see [15, Theorem 3]). We conclude by constructing various examples in §7.

We shall use the notation and definitions of [7] and [8]. We always assume that a topological space is nonempty, and that a simplex space has at least one nonzero vector. We recall that if A is a simplex space, there is a continuous one-to-one map of $EP_1(A) - \{0\}$ onto $\max A$ defined by $p \rightarrow \{p\}^\perp$. We will write $M_A(p)$ or M_p for the latter ideal, and $p(M)$ or p_M for the pre-image of a maximal ideal M . Regarding the elements of A as functions on both $EP_1(A) - \{0\}$ and $\max A$, we have $a(M) = a(p_M)$. These functions define the topology on $EP_1(A) - \{0\}$. The closure of $EP_1(A)$ is denoted by Z_A .

If Q is a closed face containing 0 in $P_1(A)$, we let

$$R(Q) = \{\alpha p : p \in E(Q), 0 \leq \alpha \leq 1\} = [0, 1]E(Q),$$

i.e., $R(Q)$ consists of the "extremal rays" in Q .

By an *isomorphism* of simplex spaces, we shall mean an isometric order isomorphism.

1. Ideals and quotients. Let $I \neq A$ be a closed ideal in a simplex space A , and $\theta: A \rightarrow A/I$ be the quotient map. Then the adjoint map $\theta^*: (A/I)^* \rightarrow A^*$ is an isometry and a weak* homeomorphism onto I° . As we saw in [7, Theorem 3.4], θ^* is also an order isomorphism; hence it restricts to an affine isomorphism of $P_1(A/I)$ onto the face I^\perp in $P_1(A)$, and thence to a homeomorphism of $EP_1(A/I) - \{0\}$ onto $E(I^\perp) - \{0\}$. From [7, Theorem 4.4], if $M \in \max A/I$, then $\theta^{-1}(M) \in \max A$ and $M \rightarrow \theta^{-1}(M)$ is a homeomorphism of $\max A/I$ onto $h(I)$. Furthermore the diagram

$$(1.1) \quad \begin{array}{ccc} EP_1(A/I) - \{0\} & \xrightarrow{\theta^*} & E(I^\perp) - \{0\} \subseteq EP_1(A) - \{0\} \\ \downarrow M_{A/I} & & \downarrow M_A \\ \max A/I & \xrightarrow{\theta^{-1}} & h(I) \subseteq \max A \end{array}$$

is commutative. We shall often identify $\max A/I$ and $h(I)$ as topological spaces. If we regard the elements of A/I as functions on $\max A/I$, they may be identified with the restrictions of elements of A to the closed set $h(I)$.

Let $J \neq \{0\}$ be a closed ideal in a simplex space A . From [7, Theorem 4.4], the restriction map $\rho: A^* \rightarrow J^*$ restricts to a continuous affine map ρ_1 of $P_1(A)$ onto

$P_1(J)$, and thence to a continuous one-to-one map ρ_E of $EP_1(A) - J^\perp$ onto $EP_1(J) - \{0\}$. If $M \in \max A - h(J)$, then $\eta_A(M) = M \cap J \in \max J$, and η_A is a homeomorphism of $\max A - h(J)$ onto $\max J$. The diagram

$$(1.2) \quad \begin{array}{ccc} EP_1(A) - J^\perp & \xrightarrow{\rho_E} & EP_1(J) - \{0\} \\ \downarrow M_A & & \downarrow M_J \\ \max A - h(J) & \xrightarrow{\eta_A} & \max J \end{array}$$

is commutative. We shall often identify $\max J$ and $\max A - h(J)$ as topological spaces. If we regard elements of J as functions on $\max J$, then they may be identified with the elements in A which vanish off the open set $\max A - h(J)$.

It is important to note that ρ_E apparently need not be a homeomorphism; hence the elements of J might not determine the topology on $EP_1(A) - J^\perp$. On the other hand we do have

LEMMA 1.1. *If K is a compact subset of $EP_1(J) - \{0\}$, then $\rho_E^{-1}(K)$ is compact in $EP_1(A) - J^\perp$.*

Proof. Letting $\rho_1: P_1(A) \rightarrow P_1(J)$ be the restriction map, it suffices to show that

$$\rho_1^{-1}(EP_1(J) - \{0\}) \subseteq EP_1(A) - J^\perp$$

since then $\rho_E^{-1}(K) = \rho_1^{-1}(K)$ will be closed and thus compact in $P_1(A)$.

If $q \in EP_1(J) - \{0\}$, the set $Q = \rho_1^{-1}(q)$ is a closed nonempty face in $P_1(A)$ disjoint from the face J^\perp . If Q did not consist of a single extreme point, $E(Q)$ would contain more than one point. Since $E(Q) \subseteq EP_1(A) - J^\perp$, this would contradict the fact that ρ_1 is one-to-one on the latter set. Thus $\rho_1^{-1}(q) = \{p\}$, where $p \in EP_1(A) - J$.

If I and J are closed ideals in a simplex space A , then Davies has shown [4] that $J + I$ is again a closed ideal. We also have

LEMMA 1.2. *Suppose that $I \neq A$, $J \neq \{0\}$, and that θ_I is the quotient map of A onto A/I . Then the relative and quotient norms and orderings coincide on $\theta_I(J) = J + I/I$, and the latter is a closed ideal in A/I . Let us identify $\max J + I/I$ with an open subset of $\max A/I$, and then identify $\max A/I$ and $\max J$ with closed and open subsets of $\max A$, respectively. When regarded as subsets of $\max A$,*

$$(1.3) \quad \max J + I/I = \max J \cap \max A/I.$$

Proof. The remark concerning norms is trivial. The relative ordering is defined by the cone

$$\begin{aligned} J + I/I \cap (A/I)^+ &= J + I/I \cap A^+/I \\ &= (J + I) \cap A^+/I = (J + I)^+/I, \end{aligned}$$

and the latter is the quotient cone. Since $J+I/I$ is a simplex space [7, Theorem 3.4], it follows that $J+I/I$ is closed in A/I , and positively generated. It is an ideal, for suppose that

$$0 \leq a+I \leq b+I, \quad b+I \in J+I/I.$$

Then letting $u, v \in I$ be such that $0 \leq a+u, a \leq b+v$, we have

$$0 \leq a+u \leq b+u+v \in J+I,$$

hence $a+u \in J+I$, and $a+I \in J+I/I$.

We have

$$\begin{aligned} \max J+I/I &= \{M \in \max A/I : M \not\supseteq J+I/I\} \\ &= \{M \in \max A : M \not\supseteq J+I, M \supseteq I\} \\ &= \{M \in \max A : M \not\supseteq J, M \supseteq I\} \\ &= \max J \cap \max A/I. \end{aligned}$$

We have a natural isomorphism

$$(1.4) \quad J+I/I \rightarrow J/I \cap J$$

defined by $a+I \rightarrow a+I \cap J$ for $a \in J$. Noting that elements of J vanish outside of $\max J$,

$$\begin{aligned} \|a+I\| &= \sup \{|a(M)| : M \in \max A/I\} \\ &= \sup \{|a(M)| : M \in \max J \cap \max A/I\} \\ &= \sup \{|a(M)| : M \in \max J - \max I\} \\ &= \sup \{|a(M)| : M \in \max J/I \cap J\} = \|a+I \cap J\|, \end{aligned}$$

hence (1.4) is an isometry. The same identifications show that $a+I \geq 0$ if and only if $a+I \cap J \geq 0$.

2. C- and M-ideals. A simplex space is a *C-space* if it is isomorphic as an ordered Banach space to $C_0(X)$, the real continuous functions vanishing at ∞ on a locally compact Hausdorff space X . For simplex spaces with identity, the following result is essentially due to H. Bauer [16].

PROPOSITION 2.1. *If A is a simplex space, the following conditions are equivalent:*

- (1) A is a *C-space*.
- (2) The elements of A are continuous on $\max A$, i.e., M_A is a homeomorphism.
- (3) $EP_1(A)$ is closed in $P_1(A)$.
- (4) $\max A$ is locally compact Hausdorff, and $A = C_0(\max A)$.

Proof. We have proved the equivalence of (2)–(4) in [8, Theorem 3.10]. (4) \Rightarrow (1) is trivial. To prove (1) \Rightarrow (3), suppose that $A = C_0(X)$, X locally compact Hausdorff. Let $X_\infty = X \cup \{\infty\}$ be the one-point compactification of X . A may be identified with the functions in $C(X_\infty)$ vanishing at ∞ . We define a continuous map δ of

X_∞ into $P_1(A)$ by $\delta(x)(a) = a(x)$. Letting ∂A be the $x \in X_\infty$ for which $\|\delta(x)\| = 0$ or 1, we have from [8, Lemma 4.1] that $\delta(\partial A) = EP_1(A)$. If $x \in X$, we may find $a \in A$ with $\|a\| \leq 1$ and $a(x) = 1$. Thus $\partial A = X_\infty$, and $EP_1(A)$ is a continuous image of a compact set.

A *C-ideal* in a simplex space A is a closed ideal which is itself a *C-space* or $\{0\}$. A point x in a topological space X is *separated* if for any $y \neq x$ there are open disjoint neighborhoods G and H of x and y , respectively. Equivalently (see [10, p. 67]), if a net x_ν in X converges to x then it cannot converge to any other point. A subset S of X is *separated* in X if all of its points are separated in X . It is readily verified that if S is such a set, compact subsets of S are closed in X . Identifying $\max J$ with the open set $\max A - h(J)$,

THEOREM 2.2. *Let $J \neq \{0\}$ be a closed ideal in a simplex space A . If J is a *C-ideal*, then*

- (1) *Compact subsets of $\max J$ are closed in $\max A$.*
- (2) *$\max J$ is separated in $\max A$.*
- (3) *Elements of J are continuous on $\max A$.*
- (4) *Elements of A restrict to continuous functions on $\max J$.*
- (5) *$Z_A \subseteq EP_1(A) \cup J^\perp$.*

*Conversely if J satisfies any one of conditions (3)–(5), it is a *C-ideal*.*

To prove this, we need

LEMMA 2.3. *Let A be a simplex space, and suppose that a net $p_\nu \in EP_1(A)$ converges to a point $q \in P_1(A)$. If Q is the smallest closed face containing q and 0 in $P_1(A)$, then M_{p_ν} converges to M_p for all $p \in E(Q) - \{0\}$.*

Proof. If $p \in E(Q) - \{0\}$, and M_{p_ν} does not converge to M_p , then there is a closed set $F \subseteq \max A$ such that $M_{p_\nu} \in F$ frequently, but $M_p \notin F$. If we let R be the closed face in $P_1(A)$ containing 0 with $F = M_A(E(R) - \{0\})$, then $p_\nu \in R$ frequently; hence $q \in R$ and $Q \subseteq R$. This contradicts the fact that $p \notin E(R) = R \cap EP_1(A)$.

Proof of Theorem 2.2. Referring to diagram (1.2) we are given that M_J is a homeomorphism. If $K \subseteq \max J$ is compact, $M_J^{-1}(K)$ is compact, and from Lemma 1.1, $K_1 = \rho_E^{-1} M_J^{-1}(K)$ is compact in $EP_1(A) - J^\perp$. It follows from [8, Corollary 3.5] that $\eta_A^{-1}(K) = M_A(K_1)$ is closed in $\max A - h(J)$.

In the remainder of the proof, we identify $\max J$ and $\max A - h(J)$. To prove (2), suppose that $M_1 \in \max J$, $M_2 \in \max A$, and $M_1 \neq M_2$. If $M_2 \in \max J$, we may separate M_1 and M_2 since $\max J$ is locally compact Hausdorff ((4) of Proposition 2.1). If $M_2 \notin \max J$, let G be an open neighborhood of M_1 with compact relative closure in $\max J$. From (1), the relative closure must be closed in $\max A$. Thus G and the complement of its closure separate M_1 and M_2 .

To prove (3), suppose that $u \in J$ and $M_\nu \rightarrow M$ is a convergent net in $\max A$. If $M \in \max J$, then eventually $M_\nu \in \max J$, and $u(M_\nu) \rightarrow u(M)$ follows from (4) of Proposition 2.1. If $M \notin \max J$, then $u(M) = 0$. Given $\varepsilon > 0$, the set

$$K = \{N \in \max A : |u(N)| \geq \varepsilon\}$$

is a compact subset of $\max J$ [7, Proposition 4.5]; hence, from (1), it is closed in $\max A$. Since $M \notin K$, we must eventually have $M_v \notin K$, hence $|u(M_v)| < \varepsilon$. We conclude that $u(M_v) \rightarrow u(M)$.

Turning to (4), it suffices to prove that if $a \in A$, and G is an open subset with compact relative closure in $\max J$, then a is continuous on G . Let $u \in J = C_0(\max J)$ be such that $u(M) = 1$ for $M \in G$. u exists since $\max J$ is locally compact. From (3), u is continuous on $\max A$; hence $ua \in J$ [8, Theorem 2.1]. Since $ua(M) = 0$ for $M \notin \max J$, $ua \in J$, and ua is also continuous on $\max A$. Since $ua(M) = a(M)$ for $M \in G$, a is continuous on G .

To prove (5), suppose that p_v is a net in $EP_1(A)$ converging to $q \in P_1(A)$, and that $q \notin J^\perp$. Let Q be the smallest closed face in $P_1(A)$ containing q and 0. Since $Q \not\subseteq J^\perp$, we may choose

$$p_0 \in E(Q) - J^\perp \subseteq EP_1(A) - J^\perp.$$

From Lemma 2.3, if $p \in E(Q) - \{0\}$, then $M_{p_v} \rightarrow M_p$, and in particular $M_{p_v} \rightarrow M_{p_0} \in \max J$. Since M_{p_0} is separated in $\max A$, $E(Q) - \{0\} = \{p_0\}$ and Q is the line segment joining 0 and p_0 . Let $q = \alpha p_0$, $0 < \alpha \leq 1$, and choose $u \in J$ with $u(M_{p_0}) \neq 0$. Then

$$u(M_{p_v}) = p_v(u) \rightarrow q(u) = \alpha p_0(u),$$

and since u is continuous on $\max J$,

$$u(M_{p_v}) \rightarrow u(M_{p_0}) = p_0(u).$$

Thus $\alpha = 1$, and $q = p_0 \in EP_1(A) - J^\perp$.

That (3) and (4) imply J is a C -ideal is apparent from (2) of Proposition 2.1. Since $\rho_1(EP_1(A)) = EP_1(J)$ and $Z_A = [EP_1(A)]^-$ is compact, we have from (5),

$$[EP_1(J)]^- = \rho_1(Z_A) \subseteq \rho_1(EP_1(A)) \cup \rho_1(J^\perp) = EP_1(J),$$

and (3) of Proposition 2.1 implies that J is a C -ideal.

At this point it would seem reasonable to define an " M -ideal" in a simplex space to be a closed ideal which is itself an M -space or $\{0\}$. However, since the analogue of Theorem 2.2 is not true for such ideals (see §7.1), it is necessary to consider a "relative" property. We say that a closed ideal J in a simplex space A is an M -ideal if $J = \{0\}$, or J satisfies one of the conditions of Theorem 2.5 (see below).

Let $F(\max A)$ be the uniformly bounded functions on $\max A$ with the uniform norm. $F(\max A)$ is an algebra and a lattice ordered Banach space (in fact, a C -space) under the usual operations. As in [8, §2], we may identify A as a Banach space with a closed subspace A' of $F(\max A)$, since

$$\|a\| = \sup \{|a(M)| : M \in \max A\}$$

(this is a simple consequence of the Krein-Milman Theorem—see the proof of

[7, Proposition 4.5]). The ordering on A coincides with the relative ordering on A' since

$$\begin{aligned} A^+ &= \{a \in A : p(a) \geq 0 \text{ for all } p \in P_1(A)\} \\ &= \{a \in A : p(a) \geq 0 \text{ for all } p \in EP_1(A) - \{0\}\} \\ &= \{a \in A : a(M) \geq 0 \text{ for all } M \in \max A\}. \end{aligned}$$

The first equality follows from [7, Theorem 2.2]. The second is due to the fact that if α is the least value a assumes on $P_1(A)$,

$$Q = \{p \in P_1(A) : p(a) = \alpha\}$$

is a closed nonempty face in $P_1(A)$; hence if $p(a) < 0$ for some $p \in P_1(A)$, the same is true for some $p' \in EP_1(A)$.

When we use the symbols \wedge and \vee below, we are taking minima and maxima in the lattice $F(\max A)$. We let $M(A)$ be the elements $a \in A^+$ such that for all $b \in A^+$, $a \wedge b \in A$. We recall that $RP_1(A)$ consists of the extremal rays of $P_1(A)$ containing 0, i.e., the line segments joining 0 to extreme points.

The following is proved in [2, pp. 99–100].

LEMMA 2.4. *Let K be a compact convex set in a locally convex space. If Q is a closed face in K and μ is a probability measure with resultant $r(\mu)$ in Q , then $\text{supp } \mu \subseteq Q$.*

THEOREM 2.5. *Suppose that $J \neq \{0\}$ is a closed ideal in a simplex space A . Then the following are equivalent:*

- (1) $J^+ \subseteq M(A)$.
- (2) $\max J$ is separated in $\max A$.
- (3) $Z_A \subseteq RP_1(A) \cup J^\perp$.

Proof. (1) \Rightarrow (2). Suppose that $M_1 \in \max J$ (the latter regarded as a subset of $\max A$) and that $M_2 \in \max A$, $M_1 \neq M_2$. Choose $u \in J^+$ with $u(M_1) \neq 0$ and $u(M_2) = 0$. This exists since if $M_2 \in \max J$, $M_1 \cap J$ and $M_2 \cap J$ are distinct maximal ideals in J ; hence we may select $u \in J \cap M_2 - J \cap M_1$. We may assume $u \geq 0$ since $J \cap M_2$ is a closed ideal, and thus is positively generated. If $M_2 \notin \max J$, we may take any $u \in J^+ - M_1$. Similarly, let $a \in A^+$ be such that $a(M_1) = 0$, $a(M_2) \neq 0$. Since $u \wedge a \in A$, the sets

$$\begin{aligned} \{M : u(M) > a(M)\} &= \{M : (u - u \wedge a)(M) > 0\}, \\ \{M : a(M) > u(M)\} &= \{M : (a - u \wedge a)(M) > 0\}, \end{aligned}$$

are open [7, Proposition 4.7], and separate M_1 and M_2 .

(2) \Rightarrow (3). Suppose that p_ν is a net in $EP_1(A)$ converging to $q \in P_1(A)$, and that $q \notin J^\perp$. Letting Q be the smallest closed face in $P_1(A)$ containing q and 0, we have Q is the line segment joining 0 to an element $p_0 \in EP_1(A) - \{0\}$ (see the proof of (5) in Theorem 2.2). Thus $q \in RP_1(A)$.

(3) \Rightarrow (1). From [8, Corollary 2.5], we may identify A with $\mathcal{A}_0(Z_A)$, the affine continuous functions on Z_A , vanishing at 0. If $u \in J^+$, and $a \in A^+$, consider the function $f(p) = \min \{u(p), a(p)\}$ defined on Z_A . f is continuous and vanishes at 0. To show it is affine, say that $p \in Z_A$ and μ_p is the extremal measure with resultant p . If $p \in J^\perp$, then from Lemma 2.4 the support of μ_p is contained in the closed face J^\perp . Since u is zero on J^\perp , and $a \geq 0$,

$$\mu_p(f) = 0 = f(p).$$

If $p \notin J^\perp$, we have $p = \alpha q$, where $q \in EP_1(A) - \{0\}$, $0 < \alpha \leq 1$. Then $\mu_p = \alpha \delta_q + (1 - \alpha) \delta_0$, and since $u, a \in \mathcal{A}_0(Z_A)$,

$$\begin{aligned} \mu_p(f) &= \alpha f(q) = \alpha \min \{u(q), a(q)\} \\ &= \min \{\alpha u(q), \alpha a(q)\} = \min \{\mu_p(u), \mu_p(a)\} \\ &= \min \{u(p), a(p)\} = f(p). \end{aligned}$$

It follows that $f \in A$, and if $M \in \max A$,

$$f(M) = f(p_M) = \min \{u(p_M), a(p_M)\} = u \wedge a(M).$$

If $u \in M(A)$, one cannot conclude that $u \wedge a \in A$ for nonpositive $a \in A$. P. Taylor has shown that if such an element u exists, then A is an M -space.

It is clear from (3) of Theorem 2.5 and Rogalski's characterization of M -spaces (see [15, Theorem 2]) that a simplex space is an M -ideal in itself if and only if it is an M -space. Thus we may delete the condition of separability in [8, Theorem 3.8] (this result has been independently obtained by F. Perdrizet).

COROLLARY 2.6. *If A is a simplex space the following conditions are equivalent:*

- (1) A is an M -space.
- (2) $\max A$ is Hausdorff.
- (3) $Z_A \subseteq RP_1(A)$.

The following technical results will play an important role in Theorem 5.2. If A is a simplex space, define $\bar{M}_A: RP_1(A) - \{0\} \rightarrow \max A$ by

$$\bar{M}_A(p) = M_A(p/\|p\|).$$

Let $I \neq A$ and $J \neq \{0\}$ be closed ideals in A . Then the following diagrams commute:

$$(2.1) \quad \begin{array}{ccc} RP_1(A/I) - \{0\} & \xrightarrow{\theta^*} & R(I^\perp) - \{0\} \subseteq RP_1(A) - \{0\} \\ \downarrow \bar{M}_{A/I} & & \downarrow \bar{M}_A \\ \max A/I & \xrightarrow{\theta^{-1}} & h(I) \subseteq \max A \end{array}$$

$$(2.2) \quad \begin{array}{ccc} RP_1(A) - J^\perp & \xrightarrow{\rho_R} & RP_1(J) - \{0\} \\ \downarrow \bar{M}_A & & \downarrow \bar{M}_J \\ \max A - h(J) & \xrightarrow{\eta_A} & \max J \end{array}$$

where ρ_R is the restriction of $\rho_1: P_1(A) \rightarrow P_1(J)$. Since ρ_1 restricts to a one-to-one map on $EP_1(A) - J^\perp$, the same is true on $RP_1(A) - J^\perp$. Again it is not clear that ρ_R is a homeomorphism, but we have the following analogue of Lemma 1.1.

LEMMA 2.7. *Suppose that J is an M -ideal in a simplex space A . If K is a compact subset of $RP_1(J) - \{0\}$, then $\rho_R^{-1}(K)$ is compact in $RP_1(A) - J^\perp$.*

Proof. Since J is an M -ideal, upon taking closures in $P_1(A)$ we have

$$[EP_1(A)]^- \subseteq RP_1(A) \cup J^\perp.$$

$[0, 1][EP_1(A)]^-$ is compact, hence closed in $P_1(A)$, and

$$[RP_1(A)]^- \subseteq [0, 1][EP_1(A)]^- \subseteq RP_1(A) \cup J^\perp.$$

It follows that

$$(2.3) \quad [\rho_R^{-1}(K)]^- \subseteq RP_1(A) \cup J^\perp.$$

But ρ_1 is continuous on $P_1(A)$ and K is closed in $P_1(J)$; hence

$$(2.4) \quad \rho_1([\rho_R^{-1}(K)]^-) \subseteq K.$$

Since $0 \notin K$, we conclude from (2.3) that

$$[\rho_R^{-1}(K)]^- \subseteq RP_1(A) - J^\perp.$$

$\rho_R = \rho_1|_{RP_1(A) - J^\perp}$ is a bijection, and we have from (2.4) that $\rho_R^{-1}(K) = [\rho_R^{-1}(K)]^-$; hence $\rho_R^{-1}(K)$ is compact.

3. Composition series of open sets. We must consider various properties that open subsets G of a topological space X can satisfy, such as

$M: G$ is separated in X .

$MLC: G$ is separated in X and locally compact in the relative topology.

If $X = \max A$ with A a simplex space, we may reformulate M (see (2) of Theorem 2.5).

$M: G = \max J$ where J is an M -ideal in A .

We also have the property

$C: G = \max J$ where J is a C -ideal in A .

In order to make our discussion precise, we define a *property* or *topological property* P to be a family of topological spaces \mathcal{P} , together with a collection P_X of open subsets for each $X \in \mathcal{P}$. We will assume that if $X \in \mathcal{P}$, then with their relative topologies, the nonempty open subsets and the nonempty closed subsets of X also lie in \mathcal{P} . If $G \in P_X$, we write $G \subseteq_P X$ and say that G is a P -set in X . If $X \subseteq_P X$, we say that X is a P -space. For M and MLC we let \mathcal{P} be all topological spaces, and for C , \mathcal{P} is the locally closed subsets of a given structure space (see §4).

If P is a property, we say that it is *inductive* if for $X \in \mathcal{P}$ and $F \neq \emptyset$ closed in X ,

I_1 . $G \subseteq_P X$ implies that $G \cap F \subseteq_P F$.

It is clear that M and MLC are inductive, and we shall prove the same for C in §4.

Given a property P and a space $X \in \mathcal{P}$, we say that an open subset G of X is *GP* in X , written $G \subseteq_{GP} X$, if for each closed subset F of X , either $G \cap F = \emptyset$ or $G \cap F$ contains a nonempty relatively open P -subset of F . *GP* is again a property, and we will use the terminology “generalized P ” or “almost P ”. From above, X is a *GP-space* if $X \subseteq_{GP} X$. We say X is an *NP-space* (or “nowhere P ”) if it contains no nonempty open P -set.

PROPOSITION 3.1. *Let P be an inductive property. If $X \in \mathcal{P}$, there is a largest \mathbf{GP} open subset G in X , and $X - G$ is *NP* or empty. Furthermore there is a collection of distinct open subsets G_γ indexed by ordinals $0 \leq \gamma \leq \gamma_0$, such that*

- (1) $G_0 = \emptyset$, $G_{\gamma_0} = G$.
- (2) If $\gamma < \gamma_0$, then G_γ is a proper subset of $G_{\gamma+1}$ and $G_{\gamma+1} - G_\gamma \subseteq_P X - G_\gamma$.
- (3) If $\gamma \leq \gamma_0$ is a limit ordinal, then $G_\gamma = \bigcup_{\beta < \gamma} G_\beta$.

Proof. Given X , let $G_0 = \emptyset$. Suppose that we have defined open G_β for ordinals $\beta < \gamma$. If γ is a limit ordinal, let $G_\gamma = \bigcup_{\beta < \gamma} G_\beta$. If γ is not a limit ordinal, let $G_\gamma = G_{\gamma-1}$ if $G_{\gamma-1} = X$ or $X - G_{\gamma-1}$ is *NP*. Otherwise let $G'_\gamma \neq \emptyset$ be a relatively open P -set in $X - G_{\gamma-1}$, and let $G_\gamma = G'_\gamma \cup G_{\gamma-1}$. Let γ_0 be the first ordinal with $G_{\gamma_0} = G_{\gamma_0+1}$, and let $G = G_{\gamma_0}$. Then $X - G$ is *NP* or \emptyset . It is clear that the G_γ satisfy (2) and (3).

Since $G_0 = \emptyset$, $G_0 \subseteq_{GP} X$. Suppose that we have proved $G_\beta \subseteq_{GP} X$ for all $\beta < \gamma$, and that F is closed in X with $F \cap G_\gamma \neq \emptyset$. Letting $\beta_0 \leq \gamma$ be the first ordinal with $F \cap G_{\beta_0} \neq \emptyset$, it is clear from (3) that β_0 is not a limit ordinal. From (2) and I_1 , $F \cap (G_{\beta_0} - G_{\beta_0-1}) = F \cap G_{\beta_0}$ is a P -set in $F \cap (X - G_{\beta_0-1}) = F$. Thus $G_\gamma \subseteq_{GP} X$.

Suppose that H is a *GP* open set in X . If $H \not\subseteq G$, then $H \cap (X - G)$ must contain a nonempty, relatively open P -subset of $X - G$. This contradicts the fact that $X - G$ is *NP*.

We will refer to a sequence G_γ having the above properties as a *P-composition series*.

If P is inductive, then letting $X = G$ in I_1 , we have that nonempty closed subsets of a P -space are P -spaces. Since for any property P , *GP* is inductive (see below), closed subsets of *GP*-spaces are again *GP*-spaces. To obtain an analogous argument for open sets, we say that a property P is *strongly inductive* if given $X \in \mathcal{P}$, G_1, G_2 open and F closed in X , it satisfies I_1 and

- I_2 . $G_1 \subseteq G_2 \subseteq X$ and $G_1 \subseteq_P X$ imply $G_1 \subseteq_P G_2$.
- I_3 . $G_1 \subseteq G_2 \subseteq_P X$ implies $G_1 \subseteq_P X$.
- I_4 . $G_1 \subseteq_P F \subseteq X$ implies $G_1 \subseteq_P X$.

It is readily verified that *M* and *MLC* are strongly inductive. We shall prove the same for *C* in §4. Note that letting $G_1 = G_2$ in I_2 , a nonempty open P -set must itself be a P -space. Letting $G_2 = X$ in I_3 , it follows that a nonempty open subset of a P -space is a P -space.

LEMMA 3.2. *If P is any property, then GP is inductive. If P is strongly inductive, then so is GP .*

Proof. Let P be an arbitrary property, and suppose that $G \subseteq_{GP} X$ and F is closed in X . A closed subset of $G \cap F$ must have the form $G \cap F'$, F' closed in X , $F' \subseteq F$. Since $G \subseteq_{GP} X$, if $G \cap F' \neq \emptyset$, then $G \cap F'$ contains a nonempty relatively open P -set in F' . Thus $G \cap F \subseteq_{GP} F$.

Suppose that P is strongly inductive. If $G_1 \subseteq G_2 \subseteq X$ and $G_1 \subseteq_{GP} X$, then given F closed in G_2 , $F = F' \cap G_2$, F' closed in X , and $F \cap G_1 = F' \cap G_1$. Choose E relatively open in $F' \cap G_1$ with $\emptyset \neq E \subseteq_P F'$. We have $E \subseteq F \subseteq F'$, and F is relatively open in F' . From I_2 for P , $E \subseteq_P F$, and we conclude GP satisfies I_2 .

If $G_1 \subseteq G_2 \subseteq_{GP} X$, then since GP is inductive, $G_1 \cap F \subseteq G_2 \cap F \subseteq_{GP} F$. Thus to prove $G_1 \subseteq_{GP} X$, all we need show is that if $G_1 \neq \emptyset$, then G_1 contains a nonempty P -subset of X . Letting X_γ , $0 \leq \gamma \leq \gamma_0$ be a P -composition series for X , we have from Proposition 3.1 that $G_2 \subseteq X_{\gamma_0}$. Let $\gamma_1 \leq \gamma_0$ be the first ordinal with $G_1 \cap X_{\gamma_1} \neq \emptyset$. Then γ_1 is not a limit ordinal. From (2) of Proposition 3.1 and I_3 for P ,

$$G_1 \cap X_{\gamma_1} = G_1 \cap (X_{\gamma_1} - X_{\gamma_1-1}) \subseteq_P X - X_{\gamma_1-1}.$$

Thus from I_4 for P , $G_1 \cap X_{\gamma_1} \subseteq_P X$, and GP satisfies I_3 .

Suppose $G_1 \subseteq_{GP} F \subseteq X$, G_1 open and F closed in X . If F' is closed in X , $G_1 \cap F' \subseteq_{GP} F \cap F' \subseteq F'$. Thus to show $G_1 \subseteq_{GP} X$, it suffices to show that if $G_1 \neq \emptyset$, it contains a nonempty P -set in X . This follows from I_4 for P .

COROLLARY 3.3. *If P is strongly inductive, then open and closed subsets of a GP -space are GP -spaces.*

A topological space is said to be K_σ if it is a countable union of compact sets, and *locally* K_σ if the K_σ -open subsets form a basis. It is clear that both open and closed subsets of a locally K_σ space are locally K_σ .

LEMMA 3.4. *If A is a simplex space, then $\max A$ is locally K_σ .*

Proof. Let M_0 be a point in $\max A$, and G be an open neighborhood of M_0 . If J is the closed ideal corresponding to G , we may choose $u \in J^+$ with $u(M_0) \neq 0$. From [7, Proposition 4.5],

$$G' = \{M : u(M) \neq 0\} = \bigcup_n \{M : u(M) \geq 1/n\}$$

is a K_σ open set with $M_0 \in G' \subseteq G$.

Applying this to C -spaces, we have the well-known result that locally compact Hausdorff spaces are locally K_σ . We note that a similar proof shows that the primitive ideal space of a C^* -algebra is locally K_σ .

We recall that a topological space X is *second category* if countable intersections of dense open sets are nonempty. X is *Baire* if such intersections are dense, or equivalently, each open set is second category. We say that X is *strongly Baire* if each closed set is Baire, or equivalently, each locally closed set (i.e., the intersection of an open and a closed set) is second category in itself. In contrast to

C^* -algebras, the structure space of a simplex space need not be strongly Baire [7, §5], or locally compact [8, Theorem 4.3].

LEMMA 3.5. *Suppose that X is locally K_σ . Then the following are equivalent:*

- (1) *X is strongly Baire and almost separated (i.e., GM).*
- (2) *X is almost locally compact separated (i.e., $GMLC$).*

Proof. (1) \Rightarrow (2). If $F \neq \emptyset$ is a closed subset of X , let $G \neq \emptyset$ be a relatively open separated subset. Since F is locally K_σ , we may choose a K_σ open set $G' \neq \emptyset$ with $G' \subseteq G$. Let $G' = \bigcup C_n$, with C_n compact. Since G' is second category, there is an n_0 with $H = \text{int } C_{n_0} \neq \emptyset$. Since G is separated in F , C_{n_0} is closed in F , hence from I_4 , $H \subseteq_{MLC} C_{n_0} \subseteq F$ implies $H \subseteq_{MLC} F$.

(2) \Rightarrow (1). Since MLC is strongly inductive, we have from Corollary 3.3 that open and closed subsets of an almost locally compact separated space have the same property. Thus it suffices to show that if X is such a space, then X is second category. Let $G \neq \emptyset$ be a locally compact Hausdorff subset of X . If G_n , $n = 1, 2, \dots$ are dense open sets in X , then $G \cap G_n$ are dense open sets in G . Since any locally compact Hausdorff space is second category, it follows that $\bigcap_n G \cap G_n \neq \emptyset$, hence $\bigcap_n G_n \neq \emptyset$.

4. Composition series of closed ideals. By analogy to §3, we must consider various properties that closed ideals J in a simplex space A can satisfy, such as

$M:J$ is an M -ideal in A .

$MLC:J$ is an M -ideal in A and $\max J$ is locally compact.

$C:J$ is a C -ideal in A .

A *simplex property* is an association with each simplex space A of a family of closed ideals P_A . If $J \in P_A$ we write $J \subseteq_P A$ and say that J is a P -ideal in A . If $A \subseteq_P A$ we say that A is a P -space. We will consider only simplex properties that are *preserved under isomorphisms*, i.e., if $\theta: A \rightarrow A'$ is an isomorphism onto, then $J \subseteq_P A$ if and only if $\theta(J) \subseteq_P \theta(A)$.

We note that M and MLC are derived from corresponding topological properties. If P is any such property, and \mathcal{P} contains the structure spaces of simplexes, we obtain a simplex property by letting $J \subseteq_P A$ if $\max J \subseteq_P \max A$.

Conversely, a simplex property defines a collection of topological properties. For each simplex space A_0 , let \mathcal{P}_{A_0} be all the nonempty locally closed subsets of $\max A_0$. Each X in \mathcal{P}_{A_0} has the form $G_0 \cap F_0$ with G_0 open, F_0 closed in $\max A_0$. We let

$$A(G_0, F_0) = \theta_{I_0}(J_0) = J_0 + I_0/I_0,$$

where $I_0 \neq A_0$, $J_0 \neq \{0\}$ are the closed ideals in A_0 with $F_0 = \max A_0/I_0$, $G_0 = \max J_0$. From (1.3) we have

$$X = G_0 \cap F_0 = \max A(G_0, F_0)$$

when the latter is considered a subset of $\max A_0$. The relatively open subsets G of X have the form $\max J$, where J is a closed ideal in $A(G_0, F_0)$. We say G is a P -set in X , written $G \subseteq_P X$, if $J \subseteq_P A(G_0, F_0)$. This definition does not depend upon (G_0, F_0) since if $G_0 \cap F_0 = G_1 \cap F_1$, then there is an isomorphism of $A(G_0, F_0)$ onto $A(G_1, F_1)$ which identifies the ideals corresponding to a given relatively open subset of X .

We say that a simplex property P is *inductive* if for each simplex space A and closed ideals $I \neq A, J$

$$I'_1: J \subseteq_P A \text{ implies } J + I/I \subseteq_P A/I.$$

We say P is *strongly inductive* if in addition,

$$I'_2: J_1 \subseteq J_2 \subseteq A \text{ and } J_1 \subseteq_P A \text{ imply } J_1 \subseteq_P J_2.$$

$$I'_3: J_1 \subseteq J_2 \subseteq_P A \text{ implies } J_1 \subseteq_P A.$$

$$I'_4: J_1 \cap I = \{0\} \text{ and } J_1 + I/I \subseteq_P A/I \text{ imply } J_1 \subseteq_P A.$$

We have:

LEMMA 4.1. *A simplex property is (strongly) inductive if and only if the corresponding topological properties are (strongly) inductive.*

Proof. Let A_0 be a simplex space, and $G \subseteq G_0$ and $F \subseteq F_0$ be open and closed subsets of $\max A_0$, respectively. The open and closed subsets of $X = G_0 \cap F_0$ have the form $G \cap F_0$ and $G_0 \cap F$, respectively. On the other hand, the closed ideals in $A = A(G_0, F_0)$ have the form $J = A(G, F_0)$. We may restate I_1 of §3 for $X = G_0 \cap F_0$ as follows:

$$G \cap F_0 \subseteq_P G_0 \cap F_0 \text{ implies } G \cap F \subseteq_P G_0 \cap F,$$

or, from the definition of \subseteq_P for locally closed sets,

$$A(G, F_0) \subseteq_P A(G_0, F_0) \text{ implies } A(G, F) \subseteq_P A(G_0, F).$$

But letting $I = A(G_0 - F, F_0)$, there is a natural isomorphism of $A(G_0, F)$ onto $A/I = A(G_0, F_0)/I$, which carries $A(G, F)$ onto $J + I/I = A(G, F_0) + I/I$. Thus

$$J \subseteq_P A \text{ implies } J + I/I \subseteq_P A/I,$$

and noting that I is a general closed ideal in A , we have the equivalence of I_1 and I'_1 .

The equivalences of I_2, I'_2 and I_3, I'_3 are trivial. Letting G_0, F_0, X, G, F be as above, we may restate I'_4 :

$$G \cap F_0 \subseteq_P G_0 \cap F \subseteq G_0 \cap F_0 \text{ implies } G \cap F_0 \subseteq_P G_0 \cap F_0.$$

We have $G \cap F_0 \subseteq G_0 \cap F$ if and only if

$$[G \cap F_0] \cap [(G_0 - F) \cap F_0] = \emptyset,$$

or regarding $A(G, F_0)$ and $I = A(G_0 - F, F_0)$ as ideals in $A(G_0, F_0)$,

$$(4.1) \quad A(G, F_0) \cap I = \{0\}.$$

Assuming $G \cap F_0 \subseteq G_0 \cap F$, we have $G \cap F_0 \subseteq_P G_0 \cap F$ if and only if $G \cap F \subseteq_P G_0 \cap F$, i.e., $A(G, F) \subseteq_P A(G_0, F)$. Using the natural isomorphism of the previous paragraph, we have $G \cap F_0 \subseteq_P G_0 \cap F$ if and only if we have (4.1) and $A(G, F_0) + I/I \subseteq_P A(G_0, F_0)/I$. Letting $A = A(G_0, F_0)$ and $J = A(G, F_0)$, we have the equivalence of I_4 and I'_4 .

Given a simplex property P , we say that a closed ideal J is *GP* in a simplex space A , written $J \subseteq_{GP} A$, if for all closed ideals $I \neq A$, $J \subseteq I$ or $J + I/I$ contains a nonzero closed P -ideal in A/I . We say that A is an *NP-space* if it contains no nonzero P -ideal. In terms of the corresponding topological property, $J \subseteq_{GP} A$ if and only if $\max J \subseteq_{GP} \max A$, and A is *NP* if and only if that is the case for $\max A$. We have:

PROPOSITION 4.2. *Let P be an inductive simplex property. Then in each simplex space A there is a largest GP-ideal J . If $J \neq A$, then A/J is NP. There is a collection of distinct closed ideals J_γ , indexed by ordinals $0 \leq \gamma \leq \gamma_0$ such that*

$$(1') J_0 = \{0\}, J_{\gamma_0} = J.$$

$$(2') \text{ If } \gamma < \gamma_0, \text{ then } J_\gamma \text{ is a proper subset of } J_{\gamma+1}, \text{ and } J_{\gamma+1}/J_\gamma \subseteq_P A/J_\gamma.$$

$$(3') \text{ If } \gamma \leq \gamma_0 \text{ is a limit ordinal, then } J_\gamma = [\bigcup_{\beta < \gamma} J_\beta]^-.$$

If P is strongly inductive, then ideals and quotients of GP-spaces are again GP-spaces.

Proof. Using the corresponding topological property for locally closed subsets of $X = \max A$, let G_γ and G be open sets as described in Proposition 3.1, and J_γ , J the closed ideals in A with $G_\gamma = \max J_\gamma$, $G = \max J$. Then (1')–(3') follow from (1)–(3) and the following remarks. From the inclusion

$$G_{\gamma+1} - G_\gamma = G_{\gamma+1} \cap (X - G_\gamma) \subseteq_P X - G_\gamma,$$

we conclude that

$$A(G_{\gamma+1}, X - G_\gamma) \subseteq_P A(X, X - G_\gamma),$$

i.e., $J_{\gamma+1}/J_\gamma \subseteq_P A/J_\gamma$. Let γ be a limit ordinal. Since the G_β , $\beta < \gamma$, form an increasing sequence, the same is true for J_β , and $\bigcup_{\beta < \gamma} J_\beta = \sum_{\beta < \gamma} J_\beta$. The latter and its closure I are ideals (see [7, §3]), and

$$\max I = X - h(I) = X - \bigcap_{\beta < \gamma} h(J_\beta) = \bigcup_{\beta < \gamma} \max J_\beta = \max J_\gamma;$$

hence $I = J_\gamma$.

The remainder of the proposition follows from Corollary 3.3.

We shall refer to a sequence of closed ideals J_γ having the above properties as a *P-composition series* for A .

We say that a simplex property P is *absolute* if closed ideals are P -ideals if and only if they are themselves P -spaces, i.e., if J is a closed ideal in A ,

$$J \subseteq_P A \quad \text{if and only if} \quad J \subseteq_P J.$$

If this is not the case, we say that P is *relative*. From its definition C is absolute,

whereas M and MLC are relative (see §7.1). If P is absolute, I'_1 and I'_3 may be restated

$$J \subseteq_P J \text{ implies } J/I \cap J \subseteq_P J/I \cap J$$

and

$$J_1 \subseteq J_2 \text{ and } J_2 \subseteq_P J_2 \text{ imply } J_1 \subseteq_P J_1,$$

respectively, and both I'_2 and I'_4 are vacuous. Thus an absolute property is (strongly) inductive if and only if (ideals and) quotients of P -spaces are P -spaces. Also it is clear that if P is absolute, the same is true for GP .

PROPOSITION 4.3. *The simplex properties M , MLC , and C are strongly inductive.*

Proof. Since the topological properties M and MLC are strongly inductive, we have from Proposition 4.1 that the same is true for the corresponding simplex properties. From the above discussion, it suffices to show that if $J \neq \{0\}$ and $I \neq A$ are closed ideals in a C -space A , then J and A/I are C -spaces. But J and A/I may be identified with the functions in A vanishing off $\max J$ and the restrictions of functions in A to $\max A/I$. In both cases, elements are continuous on the structure space; hence, from (2) of Proposition 2.1, J and A/I are C -spaces.

From Propositions 4.2 and 4.3 we conclude that every simplex space contains largest GM -, $GMLC$ -, and GC -ideals, and that both ideals and quotients of GM -, $GMLC$ -, and GC -spaces are the same.

5. The main theorems. A subset of a topological space is F_σ if it is a countable union of closed subsets of the space. We will show in §§7.2 and 7.3 that neither condition (1) nor (2) of Theorem 5.1 may be omitted, even when A is a separable M -space.

THEOREM 5.1. *If A is a separable simplex space, then A is a GC -space if and only if*

(1) $EP_1(A)$ is F_σ in $P_1(A)$ (or equivalently is K_σ),

(2) *closed subsets of $\max A$ are second category in themselves.*

One may replace (2) by

(2') *$\max A$ is strongly Baire.*

Proof. Suppose that A satisfies (1) and (2). We must show that each quotient B of A contains a nonzero C -ideal. Since the maps θ^* and θ^{-1} of (1.1) are homeomorphisms, we have that $EP_1(B)$ is K_σ and $\max B$ is second category for each quotient B . Thus changing our notation, it suffices to show that if A is a simplex space with $EP_1(A)$ a K_σ set and $\max A$ second category, then A contains a nonzero C -ideal.

Let K_n be an increasing sequence of closed, and thus compact subsets of $P_1(A)$ with $EP_1(A) = \bigcup K_n$. We may assume that $0 \in K_n$ for all n . If F is a closed subset of K_n containing 0, it is dilated in the simplex $P_1(A)$; hence its closed convex hull $Q = \bar{c}(F)$ is a face in $P_1(A)$ containing 0 [8, Theorem 3.3]. Since $F = E(Q)$ (see

[14, p. 9]), $F - \{0\}$ is closed as a subset of $\max A$, and the weak* and structure topologies coincide on each set $K_n - \{0\}$. Since elements of A are weak* continuous, they restrict to structurally continuous functions on $M_A(K_n - \{0\})$. Since $\max A$ is second category, and is the union of the closed sets $M_A(K_n - \{0\})$, one of the latter must have interior $G \neq \emptyset$ in $\max A$. Letting J be the closed ideal in A with $G = \max J$, elements of A restrict to continuous functions on $\max J$, and from Theorem 2.2, $J \subseteq_c A$.

Conversely, suppose that A is a GC -space. Since A is separable, $P_1(A)$ is compact metric and thus separable. Let J_γ , $0 \leq \gamma \leq \gamma_0$ be a C -composition series for A . The sets J_γ^\perp form a strictly decreasing transfinite sequence of closed sets in $P_1(A)$; hence γ_0 must be countable (see [11, §19, II]). We have

$$\max A = \bigcup_{\gamma < \gamma_0} [\max J_{\gamma+1} - \max J_\gamma],$$

hence

$$\begin{aligned} EP_1(A) - \{0\} &= \bigcup_{\gamma < \gamma_0} [(EP_1(A) - J_{\gamma+1}^\perp) - (EP_1(A) - J_\gamma^\perp)] \\ &= \bigcup_{\gamma < \gamma_0} [E(J_\gamma^\perp) - E(J_{\gamma+1}^\perp)]. \end{aligned}$$

It suffices to show that $B_\gamma = E(J_\gamma^\perp) - E(J_{\gamma+1}^\perp)$ is K_σ . Let $\theta: A \rightarrow A/J_\gamma$ be the quotient map. From §1, $\theta^*: EP_1(A/J_\gamma) \rightarrow E(J_\gamma^\perp)$ is a homeomorphism; hence it suffices to show that

$$B'_\gamma = \theta^{*-1}(B_\gamma) = EP_1(A/J_\gamma) - (J_{\gamma+1}/J_\gamma)^\perp$$

is K_σ . Let $\rho_1: P_1(A/J_\gamma) \rightarrow P_1(J_{\gamma+1}/J_\gamma)$ be the restriction map, and ρ_B the restriction of ρ_1 :

$$\rho_B: B'_\gamma = EP_1(A/J_\gamma) - (J_{\gamma+1}/J_\gamma)^\perp \rightarrow EP_1(J_{\gamma+1}/J_\gamma) - \{0\}$$

(see (1.2)). $J_{\gamma+1}/J_\gamma$ is a C -space; hence $EP_1(J_{\gamma+1}/J_\gamma)$ is closed in $P_1(J_{\gamma+1}/J_\gamma)$ (see Proposition 2.1). It follows that $EP_1(J_{\gamma+1}/J_\gamma) - \{0\}$ is second countable, locally compact, and thus K_σ . From Lemma 1.1, B'_γ is also K_σ .

Since any C -ideal is an M -ideal with locally compact structure, $\max A$ is $GMLC$, and from Lemmas 3.4 and 3.5, it is strongly Baire.

THEOREM 5.2. *If A is a separable simplex space, then A is a $GMLC$ -space if and only if*

- (1) $RP_1(A)$ is F_σ in $P_1(A)$ (or equivalently, is K_σ),
- (2) closed subsets of $\max A$ are second category in themselves.

One may replace (2) by

- (2') $\max A$ is strongly Baire.

Proof. Suppose that A satisfies (1) and (2). From the same argument we used in Theorem 5.1, it suffices to show that if A is a separable simplex space with $RP_1(A)$ a K_σ set and $\max A$ second category, then A contains a nonzero, MLC -ideal.

Since $P_1(A)$ is metrizable and second countable, open subsets are F_σ , hence $RP_1(A) - \{0\}$ is K_σ . Let K_n be compact sets with $RP_1(A) - \{0\} = \bigcup K_n$. If F is a closed subset of K_n , let $D(F)$ be all $p \in P_1(A)$ such that for some $\alpha \geq 0$ and $q \in F$, $p = \alpha q$. Then $D(F)$ is clearly dilated in $P_1(A)$, and we claim that it is closed. Suppose p_ν is a net in $D(F)$ converging to $p \in P_1(A)$. Let $p_\nu = \alpha_\nu q_\nu$, $q_\nu \in F$, $\alpha_\nu \geq 0$. Choosing a subnet, we may assume that $q_\nu \rightarrow q \in F$. The sequence α_ν must be eventually bounded, since if not, $q_\nu \in \alpha_\nu^{-1}P_1(A)$ will imply $q = 0$. Choosing a subnet, we may assume $\alpha_\nu \rightarrow \alpha \geq 0$. But then $\alpha_\nu q_\nu \rightarrow \alpha q$, $p = \alpha q$, and $p \in D(F)$. From [8, Theorem 3.3], $Q = \bar{c}(D(F))$ is a closed face in $P_1(A)$ with $E(Q) \subseteq D(F)$. We have (see (2.2))

$$\bar{M}_A(F) = \bar{M}_A(D(F) - \{0\}) = M_A(E(Q) - \{0\});$$

hence $\bar{M}_A(F)$ is closed in $\max A$. We conclude that $\bar{M}_A: K_n \rightarrow \max A$ is a closed map; hence, since K_n is compact Hausdorff, $\bar{M}_A(K_n)$ is a compact Hausdorff space (see [3, p. 97]). But $\max A$ is second category and it is the union of the closed sets $\bar{M}_A(K_n)$; hence one of the latter has interior $G \neq \emptyset$ in $\max A$. G is separated in $\bar{M}_A(K_n)$ and the latter is closed in $\max A$; hence G is separated in $\max A$. It is also locally compact. Letting J be the closed ideal in A with $G = \max J$, we see that J is an *MLC*-ideal in A .

Suppose that A is a *GM*-space. $P_1(A)$ is compact metric, hence separable. Let J_γ , $0 \leq \gamma \leq \gamma_0$, be an *M*-composition series for A . As in the proof of Theorem 5.1, γ_0 must be countable, and we have

$$RP_1(A) - \{0\} = \bigcup_{\gamma < \gamma_0} R(J_\gamma^\perp) - R(J_{\gamma+1}^\perp),$$

and it suffices to show that $B_\gamma = R(J_\gamma^\perp) - R(J_{\gamma+1}^\perp)$ is K_σ . Let $\theta: A \rightarrow A/J_\gamma$ be the quotient map.

$$\theta^*: RP_1(A/J_\gamma) \rightarrow R(J_\gamma^\perp)$$

is a homeomorphism; hence it suffices to show

$$B'_\gamma = \theta^{*-1}(B_\gamma) = RP_1(A/J_\gamma) - (J_{\gamma+1}/J_\gamma)^\perp$$

is K_σ . Let $\rho_1: P_1(A/J_\gamma) \rightarrow P_1(J_{\gamma+1}/J_\gamma)$ be the restriction map, and ρ_R the restriction of ρ_1 :

$$\rho_R: B'_\gamma = RP_1(A/J_\gamma) - (J_{\gamma+1}/J_\gamma)^\perp \rightarrow RP_1(J_{\gamma+1}/J_\gamma) - \{0\}$$

(see (2.2)). Since $J_{\gamma+1}/J_\gamma$ is an *M*-space, $RP_1(J_{\gamma+1}/J_\gamma)$ is closed in $P_1(J_{\gamma+1}/J_\gamma)$. It follows that $RP_1(J_{\gamma+1}/J_\gamma) - \{0\}$ is second countable locally compact, and thus K_σ . Since $J_{\gamma+1}/J_\gamma$ is an *M*-ideal in A/J_γ , we have from Lemma 2.7 that B'_γ is also K_σ .

If A is *GMLC*, then from Lemmas 3.4 and 3.5, $\max A$ is strongly Baire.

Note that we proved that if A is separable, then A is a *GM*-space implies $RP_1(A)$ is K_σ . By using the example in [7, §5] one can easily show the converse is false.

THEOREM 5.3. *If A is a *GM*-space, then the following are equivalent:*

- (1) $\max A$ is almost locally compact Hausdorff (i.e., A is a *GMLC*-space).

(2) $\max A$ is strongly Baire.

(3) If $a \in A$, then for each closed subset F of $\max A$, $a|F$ has a point of continuity in each subset of F that is relatively open and separated in F .

Proof. (1) \Rightarrow (2). This follows from Lemmas 3.4 and 3.5.

(2) \Rightarrow (3). If I is a proper closed ideal in A with $F = \max A/I$, we may identify the functions $a|F$ with the elements of A/I . In addition, closed subsets of $\max A$ are Baire. Thus it suffices to show that if A is a simplex space with Baire structure space, then each $a \in A$ has a point of continuity in each nonempty separated open subset G . Since G is second category, it will suffice to show that $f = a|G$ is of first Baire class, i.e., if S is a closed subset of the reals, then $f^{-1}(S)$ is a G_δ subset of G (see [11, §27, X]). The difference of two first Baire class functions is again the same (see [11, §27, VI]) and $A = A^+ - A^+$; hence we may assume $a \geq 0$.

It will suffice to show that f is upper semicontinuous on G , i.e., given $\alpha > 0$, the set

$$F = \{M \in G : f(M) \geq \alpha\} = G \cap \{M \in \max A : a(M) \geq \alpha\}$$

is relatively closed in G . Suppose that $M_\nu \in F$ converges to $M \in G$. Since $C = \{M \in \max A : a(M) \geq \alpha\}$ is compact [7, Proposition 4.5], taking a subnet we may assume that M_ν converges to $N \in C$. Since G is separated in $\max A$, $M = N$ and $M \in F$.

(3) \Rightarrow (1). It suffices to show that if A is GM and each $a \in A$ has a point of continuity in each nonempty open separated subset, then $\max A$ contains a nonempty locally compact separated open subset. Since A is GM , there is a separated open subset G in $\max A$. Let $G = \max J$, J a closed ideal in A , and choose $a \in J^+$, $a \neq 0$. The set

$$G_1 = \{M \in \max A : a(M) > 0\}$$

is an open subset of G [7, Proposition 4.7], and thus is separated in $\max A$. Let $M_0 \in G_1$ be a point of continuity of a . Then letting $a(M_0) = \varepsilon > 0$, the set

$$C = \{M \in \max A : a(M) \geq \varepsilon/2\}$$

must have nonempty interior G_2 with $M_0 \in G_2$. On the other hand G_2 is an open subset of the compact Hausdorff space C , and thus is locally compact. G_2 is separated in $\max A$ since $G_2 \subseteq G$.

COROLLARY 5.4. *If A is an M -space, then the following are equivalent:*

(1) $\max A$ is almost locally compact.

(2) $\max A$ is strongly Baire.

(3) If $a \in A$, then for each closed subset F of $\max A$, the points of continuity of $a|F$ are dense in F .

It might at first seem rather difficult to determine whether $EP_1(A)$ or $RP_1(A)$ are F_σ in $P_1(A)$. That this is often not the case is due to a result in point set topology.

If S is a subset of a topological space X , define the *residue* $\rho(S)$ of S in X by $\rho(S) = [\bar{S} - S]^- \cap S$. We have (see [11, §30, VI]):

PROPOSITION 5.5. *If X is a complete metric space, then a subset S of X is both G_δ and F_σ if and only if for each nonempty relatively closed subset F of S , $\rho(F)$ is a proper subset of F .*

If A is separable, this is applicable to $EP_1(A)$, since then $P_1(A)$ is metrizable, and $EP_1(A)$ is already G_δ (see [14, Proposition 1.3]). The same is true for $RP_1(A)$ since

LEMMA 5.6. *If A is a separable simplex space, then $RP_1(A)$ is G_δ in $P_1(A)$.*

Proof. We say that $q, r \in P_1(A)$ are *proportional* if there is an $\alpha \in [0, 1]$ with $q = \alpha r$ or $r = \alpha q$. We have $p \notin RP_1(A)$ if and only if there exist nonproportional $q, r \in P_1(A)$ such that $p = \frac{1}{2}q + \frac{1}{2}r$. To show this, note $p \notin RP_1(A)$ if and only if $p \neq 0$ and

$$p/\|p\| \notin EP_1(A) - \{0\} = ES(A).$$

But the latter will be true if and only if $p/\|p\|$ is not the midpoint of two distinct elements of $S(A)$.

Define $F_1, F_2 \subseteq P_1(A) \times P_1(A)$ by

$$F_1 = \{(q, r) : q = \alpha r, \alpha \in [0, 1]\},$$

$$F_2 = \{(q, r) : \alpha q = r, \alpha \in [0, 1]\},$$

and $\psi: P_1(A) \times P_1(A) \rightarrow P_1(A)$ by

$$\psi((p, q)) = \frac{1}{2}p + \frac{1}{2}q.$$

We have $RP_1(A) = P_1(A) - \psi(G)$, where $G = P_1(A) \times P_1(A) - F_1 \cup F_2$. Since the sets F_i are closed, G is open in the compact metric space $P_1(A) \times P_1(A)$, and thus is K_σ . Since ψ is continuous, $\psi(G)$ is K_σ , and $RP_1(A)$ is G_δ .

We note that if $EP_1(A)$ is K_σ , then so is $RP_1(A) = [0, 1]EP_1(A)$, since products of compact sets are the same. The converse is false (see §7.2).

For a typical application of Proposition 5.5, let $P_\alpha(A)$, $\alpha \geq 0$, be the positive functions p satisfying $\|p\| \leq \alpha$.

PROPOSITION 5.7. *Let A be a separable simplex space. If for some $\alpha < 1$ we have*

$$Z_A = [EP_1(A)]^- \subseteq EP_1(A) \cup P_\alpha(A),$$

then $EP_1(A)$ is F_σ in $P_1(A)$.

Proof. Since $P_\alpha(A)$ is closed,

$$\rho(EP_1(A)) = [Z_A - EP_1(A)]^- \cap EP_1(A) \subseteq P_\alpha(A) \cap EP_1(A) = \{0\};$$

hence $\rho(\rho(EP_1(A))) = \rho(\{0\}) = \emptyset$. It follows that ρ must map nonempty relatively closed subsets of $EP_1(A)$ onto proper subsets.

It is interesting to note that if X is locally compact Hausdorff (e.g., if $X = P_1(A)$), then a subset S of X is itself locally compact if and only if it is locally closed, or equivalently, $\rho(S) = \emptyset$ (see [12, §39, VII]).

6. The Choquet boundary. We wish to prove an analogue of the Bishop-de Leeuw characterization of the Choquet boundary of a function algebra (see [14, §8]). Suppose that X is a compact Hausdorff space and that A is a linear subspace of $C(X)$ that is a simplex space with the relative norm and ordering. If $x \in X$, the functional $\delta(x)$ defined on A by $\delta(x)(a) = a(x)$ is positive and of norm less than or equal to 1, i.e., $\delta(x) \in P_1(A)$. We define the *Choquet boundary* ∂A to be the points $x \in X$ such that $\delta(x) \in EP_1(A)$. We say $x \in X$ is a *semipeak point* if $\delta(x) = 0$ or for each $y \in X$ with $\delta(y) \neq \delta(x)$, there is an $a \in A$ with

$$(6.1) \quad a(y) < a(x) = \|a\|.$$

THEOREM 6.1. *The Choquet boundary ∂A consists of the semipeak points, and $\delta(\partial A)$ equals either $EP_1(A) - \{0\}$ or $EP_1(A)$.*

Proof. The argument given in [8, Lemma 4.1] is still valid; hence $\delta(X) \supseteq EP_1(A) - \{0\}$, and the second assertion follows.

If $\delta(x) \in EP_1(A) - \{0\}$, and $\delta(y) \neq \delta(x)$, let Q be the smallest closed face in $P_1(A)$ containing $\delta(y)$. If Q is contained in the line segment $[0, \delta(x)]$ joining 0 and $\delta(x)$, then there is an α with $0 \leq \alpha < 1$ and $\delta(y) = \alpha\delta(x)$. If $a \in A$ and $a(x) > 0$,

$$a(y) = \alpha a(x) < a(x),$$

and to prove (6.1) it suffices to select a with $a(\delta(x)) = \|a\| > 0$. The existence of such an element was proved in [8, Lemma 4.1]. If $Q \not\subseteq [0, \delta(x)]$, choose $q \in E(Q) - \{0, \delta(x)\}$. $M_q = [0, q]^+$ is a (maximal) closed ideal in A [7, Corollary 3.2], and from the proof of [7, Theorem 4.4], if ρ_1 is the restriction map,

$$\rho_1(EP_1(A) - \{0, q\}) = EP_1(M_q) - \{0\}.$$

The previously mentioned existence theorem gives us an element $a \in M_q$ with $a(p) = \|a\| \neq 0$. The set

$$R = \{r \in P_1(A) : a(r) = \|a\|\}$$

is a closed face in $P_1(A)$. If $a(\delta(y)) = a(y) = \|a\|$, then $\delta(y) \in R$, $Q \subseteq R$, and $q \in R$. This contradicts the fact that $a(q) = 0$. Thus x is a semipeak point.

Conversely, suppose that $\delta(x) \notin EP_1(A)$, and let Q be the smallest closed face in $P_1(A)$ containing $\delta(x)$. Since $E(Q) \subseteq EP_1(A)$ we may choose $y \in X$ with $\delta(y) \in E(Q) - \{0\}$. We have $\delta(y) \neq \delta(x)$. If $a \in A$, define R as above. Then if $a(x) = \|a\|$, $\delta(x) \in R$ and $Q \subseteq R$. It follows that $\delta(y) \in R$, i.e., $a(y) = \|a\|$, and inequality (6.1) cannot hold. Thus x is not a semipeak point.

In the following section, we will say that $x \in X$ is a *peak point* if there is an $a \in A$ such that $0 \leq a(y) < a(x)$ for all y with $\delta(y) \neq \delta(x)$. It is clear that a peak point must be a semipeak point.

7. Some examples.

7.1. *A closed ideal which is an M-space but not an M-ideal.* Let J be the positive integers, and $J^* = J \cup \{\infty\}$ the one-point compactification of J . Let $A \subseteq C(J^*)$ be all functions a such that

$$(7.1) \quad a(\infty) = \frac{1}{2}a(1) + \frac{1}{2}a(2),$$

together with the relative norm and ordering. A is a simplex space. (See [2, Theorem 1], [13, p. 78]. The proof simply uses the fact that the map ρ defined as in [8, Lemma 4.1] is one-to-one and onto $S(A)$.) Define ε_n on J^* for $1 \leq n \leq \infty$ by

$$\begin{aligned} \varepsilon_n(m) &= 0 & m \neq n \\ &= 1 & m = n. \end{aligned}$$

Then ε_n is in A for $3 \leq n < \infty$, as are the functions

$$f_1 = 2\varepsilon_1 + \sum_{3 \leq i \leq \infty} \varepsilon_i, \quad f_2 = 2\varepsilon_2 + \sum_{3 \leq i \leq \infty} \varepsilon_i.$$

Noting that in particular f_1 assumes distinct values at 1, 2, and ∞ , we have that A distinguishes points in J^* , hence $\delta: J^* \rightarrow P_1(A)$ is a homeomorphism. Using ε_n , $3 \leq n < \infty$, it is clear that n is a peak point for these values. 1 is a peak point since $f_1(1) > f_1(n)$ for all $n \neq 1$. Similarly 2 is a peak point. ∞ is not a peak point since from (7.1), $a(\infty)$ must lie between $a(1)$ and $a(2)$. Thus $\partial A = J$, and from Theorem 6.1,

$$EP_1(A) = \{0\} \cup \{\delta(n) : 1 \leq n < \infty\}.$$

From [7, Corollary 3.2],

$$I = \{a \in A : a(1) = 0\} = \{\delta(1)\}^\perp$$

is a (maximal) closed ideal in A since the line segment joining 0 and $\delta(1)$ is a closed face in $P_1(A)$. I may be identified with

$$\{a \in C(J^* - \{1\}) : a(\infty) = \frac{1}{2}a(2)\}.$$

Since the latter is a sublattice of $C(J^* - \{1\})$, it is an M -space. On the other hand, $n \rightarrow \infty$, hence

$$\delta(n) \rightarrow \delta(\infty) = \frac{\delta(1) + \delta(2)}{2}$$

and from Lemma 2.3, $M_{\delta(n)}$ converges to both $M_{\delta(1)}$ and $M_{\delta(2)}$. It follows that although $M_{\delta(2)} \in \max I$, it is not a separated point in $\max A$.

Although I is not a C -space, A and thus I are GC -spaces. Letting

$$I_1 = \{a \in C(J^*) : a(1) = a(2) = 0\},$$

we have that $\{0\} \subseteq I_1 \subseteq A$ is a C -composition series since I_1 may be identified with $C_0(J^* - \{1, 2, \infty\})$, and A/I_1 with $C(\{1, 2\})$.

7.2. A locally compact M (i.e., MLC) space which is nowhere C (i.e., an NC-space). We will construct a separable M -space A for which $\max A$ is compact, and

$$(7.2) \quad EP_1(A) - \{0\} \subseteq [EP_1(A)^- - EP_1(A)]^-.$$

If J is a C -ideal in A , we have from Theorem 2.2, $EP_1(A)^- \subseteq EP_1(A) \cup J^\perp$; hence

$$EP_1(A) \subseteq [EP_1(A)^- - EP_1(A)]^- \cup \{0\} \subseteq J^\perp,$$

and $J^\perp = P_1(A)$, i.e., $J = \{0\}$.

Let $C_b([0, \infty))$ be the bounded continuous real functions on $[0, \infty)$ with the uniform norm and the usual ordering. $C_b([0, \infty))$ may be identified with $C(\beta[0, \infty))$ where $\beta[0, \infty)$ is the Stone-Čech compactification of $[0, \infty)$, since each $f \in C_b([0, \infty))$ has an isometric order isomorphic extension to $\beta[0, \infty)$. Thus $C_b([0, \infty))$ is a C -space.

Let $\alpha_n, n \geq 1$ be a sequence of scalars with $0 < \alpha_n < 1$, and $\prod \alpha_n > 0$ (e.g., let $\alpha_n = 1 - 1/(n+1)^2$ —see [1, p. 155]). Let $r_n, n \geq 1$ be an enumeration of the rationals in $[0, \infty)$. We may assume that $r_n \leq n-1$, hence in particular, $r_1 = 0$. We define A_0 to be the functions a in $C_b([0, \infty))$ such that $a(n) = \alpha_n f(r_n), n \geq 1$. Although it can be shown that A_0 has the desired properties, it is not separable. We will use instead the sublattice generated by a countable subset.

Define $a_0 \in C_b([0, \infty))$ inductively: Let $a_0(0) = 1$. Having defined $a_0(x)$ for $0 \leq x \leq n$, and using the fact that $r_{n+1} \leq n$, let

$$(7.3) \quad a_0(n+1) = \alpha_{n+1} a_0(r_{n+1}),$$

and interpolate linearly between n and $n+1$. It is clear that $a_0 \in A_0$, and that a_0 is strictly decreasing. In particular a_0 distinguishes points in $[0, \infty)$. We also have that if $0 < x \leq n$,

$$a_0(x) \geq a_0(n) = \alpha_n a_0(r_n) \geq \alpha_n a_0(n-1);$$

hence, letting $\varepsilon = \prod \alpha_n$,

$$(7.4) \quad a_0 \geq \varepsilon 1.$$

To obtain peaking at points other than the integers, we define $a_k \in C_b([0, \infty))$, $k \geq 1$, inductively. Let $a_k(0) = 1$. Having defined $a_k(x)$ for $0 \leq x \leq n$, define $a_k(n+1)$ as in (7.3). Let $a_k(x) = 1$ for $x \in [n+1/2k, n+1-1/2k]$, and interpolate linearly in the intervals $[n, n+1/2k]$ and $[n+1-1/2k, n+1]$.

We let A be the smallest closed lattice subspace containing $a_k, 0 \leq k < \infty$. To see that A is separable, let B be all finite linear combinations $\sum \alpha_i a_i$ with α_i non-negative rationals. B is closed under addition and multiplication by nonnegative rationals. Let C be all elements of the form $b_1 \vee \cdots \vee b_m$ with $b_i \in B$. C is closed under unions, additions, and multiplication by nonnegative rationals, since

$$(b_1 \vee \cdots \vee b_m) + (b'_1 \vee \cdots \vee b'_n) = \bigvee_{i,j} (b_i + b'_j)$$

and

$$\alpha(b_1 \vee \cdots \vee b_m) = \alpha b_1 \vee \cdots \vee \alpha b_m.$$

$C - C$ is closed under unions since

$$(c_1 - c_2) \vee (c_3 - c_4) = (c_1 + c_4) \vee (c_3 + c_2) - (c_2 + c_4),$$

and $C - C$ is closed under addition and multiplication by all rational scalars. Since these three operations are continuous in $C_b([0, \infty))$ (see [5, p. 99] for unions), the closure $[C - C]^-$ is again closed with respect to them. It follows that $[C - C]^-$ is closed with respect to multiplication by all scalars, and is a closed vector lattice. Thus $A = [C - C]^-$, and since $C - C$ is countable, A is separable.

We will regard A as a sublattice of $C(\beta[0, \infty))$. Since $A \subseteq A_0$,

$$(7.5) \quad \delta(n) = \alpha_n \delta(r_n), \quad \alpha_n < 1, n \geq 1;$$

hence $\|\delta(n)\| < 1$, and, letting J be the positive integers,

$$\delta(J) \cap EP_1(A) - \{0\} = \emptyset.$$

If $x \in [0, \infty) - J$, then choosing integers $n \geq 0, k \geq 1$ with

$$n + 1/2k \leq x \leq n + 1 - 1/2k,$$

we have that $a_k(x) = 1$, $\|\delta(x)\| = 1$, and $\delta(x) \in EP_1(A)$; ($\delta(x)$ already lies on an extremal ray). Since $[0, \infty) - J$ is dense in $[0, \infty)$, we have

$$\delta(J) \subseteq EP_1(A)^- - EP_1(A).$$

To prove (7.2), note that $\delta(J)^- \supseteq \delta([0, \infty))$, since letting $x \in [0, \infty)$, and r_{n_k} be a subsequence of r_n with $r_{n_k} \rightarrow x$, we have $\alpha_{n_k} \rightarrow 1$ and, from (7.5),

$$\delta(n_k) = \alpha_{n_k} \delta(r_{n_k}) \rightarrow \delta(x).$$

$[0, \infty)$ is dense in $\beta[0, \infty)$; hence

$$\delta(J)^- \supseteq \delta(\beta[0, \infty)) \supseteq EP_1(A) - \{0\},$$

the latter inclusion being a consequence of Theorem 6.1.

Finally, since a_0 must vanish at ∞ [7, Proposition 4.5], we have from (7.4) that $\max A$ is compact Hausdorff, hence strongly Baire. From (7.4), $F = EP_1(A) - \{0\}$ is relatively closed in $EP_1(A)$. Taking residues (see §5), a simple argument shows that $\rho(F) = \rho(EP_1(A))$. From (7.2), $\rho(F) = F$. Using Proposition 5.5 we conclude that $EP_1(A)$ is not K_σ . Thus A satisfies (2) but not (1) of Theorem 5.1.

7.3. *An M -space which is nowhere locally compact (i.e., an NMLC-space).* We will construct a separable M -space A such that $\max A$ is countable and dense in itself, i.e., if $M \in \max A$, then $M \in [\max A - \{M\}]^-$. If G were a nonempty locally compact open subset of $\max A$, G would also be dense in itself. But G would be second category, and letting $G = \{M_1, M_2, \dots\}$, $G - \{M_i\}$ would be dense in G ; hence $\emptyset = \bigcap_i G - \{M_i\}$ would be the same, a contradiction.

Let $J^* = J \cup \{\infty\}$ be the one-point compactification of the positive integers. Let φ be a map of $J - \{1\}$ onto $J \times J$ such that if $\varphi(m) = (p, q)$, then $p < m$. Let α_m , $m \geq 2$ be any sequence with $0 < \alpha_m < 1$ and α_m decreasing to zero. We define A to be the functions $a \in C(J^* \times J^*)$ such that

$$(7.6) \quad \begin{aligned} a(1, \infty) &= 0, \\ a(m, \infty) &= \alpha_m a(\varphi(m)), \quad 2 \leq m \leq \infty, \\ a(\infty, n) &= 0, \quad 1 \leq n \leq \infty. \end{aligned}$$

If $1 \leq i, j < \infty$, define a function a_{ij} on $J^* \times J^*$ as follows:

$$\begin{aligned} a_{ij}(m, n) &= 0, \quad m < i, 1 \leq n \leq \infty, \\ a_{ij}(i, n) &= 0, \quad n \neq j, 1 \leq n \leq \infty, \\ a_{ij}(i, j) &= 1. \end{aligned}$$

For $m > i$, we proceed inductively. Suppose that $a_{ij}(m, n)$ has been defined for $i \leq m \leq m_0$, and $1 \leq n \leq \infty$. Then define

$$a_{ij}(m_0 + 1, n) = \alpha_{m_0 + 1} a_{ij}(\varphi(m_0 + 1)), \quad 1 \leq n \leq \infty.$$

Note that if $\varphi(m_0 + 1) = (p, q)$, then $p \leq m_0$, and $a_{ij}(\varphi(m_0 + 1))$ is already defined. Finally define

$$a_{ij}(\infty, n) = 0, \quad 1 \leq n \leq \infty.$$

We claim that a_{ij} is continuous. We have

$$\begin{aligned} a_{ij}(m, \infty) &= \lim_n a_{ij}(m, n) = 0, \quad 1 \leq m \leq i, \\ a_{ij}(m, \infty) &= \lim_n a_{ij}(m, n) = \alpha_m a_{ij}(\varphi(m)), \quad i < m < \infty. \end{aligned}$$

On the other hand, if $i < m < \infty$,

$$a_{ij}(m, n) \leq \alpha_m, \quad 1 \leq n \leq \infty.$$

If (m_k, n_k) is a sequence converging to (∞, n) , then for any $m_0 > i$, eventually $m_k \geq m_0$, and $a_{ij}(m_k, n_k) \leq \alpha_{m_0}$; hence

$$\lim_k a_{ij}(m_k, n_k) = a_{ij}(\infty, n) = 0.$$

It is clear that a_{ij} satisfies (7.6) and thus is an element of A .

Since $a_{ij}(i, j) = 1$ and $a_{ij}(m, n) < 1$ for $(m, n) \neq (i, j)$, we have that $J \times J \subseteq \partial A$, and A distinguishes points in $J \times J$. From (7.6),

$$\begin{aligned} \delta((1, \infty)) &= 0, \\ \delta((m, \infty)) &= \alpha_m \delta(\varphi(m)), \quad 2 \leq m < \infty, \\ \delta((\infty, n)) &= 0, \quad 1 \leq n \leq \infty; \end{aligned}$$

hence $\delta(J \times J) = EP_1(A) - \{0\}$. Given any pair (p, q) , letting $\varphi(m) = (p, q)$ we have $(m, n) \xrightarrow{n} (m, \infty)$; hence

$$\delta((m, n)) \xrightarrow{n} \delta((m, \infty)) = \alpha_m \delta((p, q)).$$

From Lemma 2.3,

$$M_{\delta((m, n))} \xrightarrow{n} M_{\delta((p, q))}$$

and $\max A$ is dense in itself.

Since $EP_1(A) - \{0\} = \delta(J \times J)$ is countable, it is K_σ , and A satisfies (1) but not (2) of Theorem 5.1. We note that as a consequence of [8, Corollary 3.9], $\max A$ is not first countable, even though it is a countable set.

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